

# Theory of thin shells in general relativity: Equivalence of direct and Wheeler-DeWitt quantizations

Konstantin G. Zloshchastiev

Department of Theoretical Physics, Dnepropetrovsk State University,  
Nauchniy lane 13, Dnepropetrovsk 320625, Ukraine.

## Abstract

We justify the way of the direct quantization which means immediate quantization of a conservation law. It is shown that this approach is equivalent to introducing the super Hamiltonian on a minisuperspace in spirit of the Wheeler-DeWitt's approach. Then we will have: all values are observable and have an obvious physical meaning and well-defined application domain; wave function is well-defined without time slicing and often can be exactly obtained; we can take off major mathematical troubles, and therefore, more complicated models can be considered exactly without the perturbation theory.

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Address for hard correspondence:

K. G. Zloshchastiev  
Metrostroevskaya 5, fl.453  
Dnepropetrovsk 320128  
Ukraine

Email:

zloshch@hotmail.com

Beginning from the classical works [1, 2] the investigation of thin shells in general relativity has got large development (see reviews [3]). In present paper we will consider the general class of spherically symmetric shells with nonzero surface tension [4] thereby main attention will be paid to quantum aspects of the theory.

Let us consider a thin layer with surface stress-energy tensor of a perfect fluid in general case (we use the units  $\gamma = c = 1$ , where  $\gamma$  is the gravitational constant)

$$S_{ab} = \sigma u_a u_b + p(u_a u_b + {}^{(3)}g_{ab}), \quad (1)$$

where  $\sigma$  and  $p$  are the surface energy density and pressure respectively,  $u^a$  is the timelike unit vector,  ${}^{(3)}g_{ab}$  is the metric on the shell.

We shall write the metrics of the spacetimes outside  $\Sigma^{out}$  and inside  $\Sigma^{in}$  the spherical shell in the form

$$ds_{(in)}^2 = -\Phi^\pm(r) dt_\pm^2 + \Phi^\pm(r)^{-1} dr^2 + r^2 d\Omega^2, \quad (2)$$

where  $d\Omega^2$  is the metric of unit 2-sphere. It is possible to show that if one uses the proper time  $\tau$  of a shell, then the energy conservation law can be written as

$$d \left( \sigma \sqrt{{}^{(3)}g} \right) = -p d \left( \sqrt{{}^{(3)}g} \right) - \sqrt{{}^{(3)}g} \left[ (T^{\tau n})^{out} - (T^{\tau n})^{in} \right] d\tau, \quad (3)$$

where  $T^{\tau n} = T^{\alpha\beta} u_\alpha n_\beta$  is the projection of stress-energy tensors in the  $\Sigma^{out}$  and  $\Sigma^{in}$  spacetimes on the tangent and normal vectors,  ${}^{(3)}g = \det({}^{(3)}g_{ab})$ . The worldsheet metric of a shell is

$${}^{(3)}ds^2 = -d\tau^2 + R^2 d\Omega^2, \quad (4)$$

where  $R(\tau)$  turns to be the proper radius of the shell.

Imposing junction conditions across the shell, we derive the equations of motion of such shells in the form

$$\begin{aligned} \epsilon_+ \sqrt{\dot{R}^2 + \Phi^+(R)} - \epsilon_- \sqrt{\dot{R}^2 + \Phi^-(R)} &= -\frac{m}{R}, \\ m &= 4\pi\sigma(R)R^2, \end{aligned} \quad (5)$$

where  $\dot{R} = dR/d\tau$ ,  $m$  is the (effective) rest mass. The choice of the pair  $\{\epsilon_+ = \pm 1, \epsilon_- = \pm 1\}$  divides all shells into the classes of black hole (BH) type and traversable wormhole (WH) type shells. Equations (3) and (5) together with the state equation  $p = p(\sigma, {}^{(3)}g)$  and choice of the signs  $\epsilon_\pm$  uniquely determine the motion of the fluid shell. For further we will assume  $\sigma(R)$  as an already known function of the theory because in most cases we can resolve the conservation law (3) independently of an equation of motion [5]. Equation (5) can also be rewritten without roots: double squaring we obtain

$$\dot{R}^2 = \left[ \frac{\Delta\Phi - m^2/R^2}{2m/R} \right]^2 - \Phi^-(R), \quad (6)$$

where  $\Delta\Phi = \Phi^+(R) - \Phi^-(R)$ .

At present there are many approaches to quantize thin shells that is connected with different way of constructing the Hamiltonian structure [6, 7, 8, 9]. Of course, (almost) all these methods give the different results (wave functions, spectra, etc), thus we can observe the non-equivalent theories in all their discouraging multiformity. Therefore it is necessary to work out some unified approach by means of which we could compare models. Besides, much of the known approaches use the perturbation theory to obtain final results that can be dangerous within the framework of highly non-linear general relativity. Thus it would be very important if this unified approach would be also maximum nonperturbative.

The pure minisuperspace approach which does not require any time slicing and time gauge seems to be the most suitable candidate (see, e.g., Ref. [10]). Indeed, if one takes a look at Eq. (6), one can see no time as a variable. Moreover, the second order differential equations from which Eq. (6) was obtained, also contain no time variable [2]. Therefore, what is the reason for introducing a time (and related concepts) forcibly, all the more so it creates additional troubles? Let us consider the two approaches within the frameworks of minisuperspace method.

(a) *Approach with effective mass*

Let us consider the minisuperspace model initially described by the Lagrangian

$$L = \frac{m\dot{R}^2}{2} - \frac{m}{2} \left\{ \Phi^- - \left[ \frac{\Delta\Phi - m^2/R^2}{2m/R} \right]^2 \right\}, \quad (7)$$

where we mean the integral as a primitive,  $m$  is the above-mentioned effective rest mass,  $m = m(R)$ . The equation of motion is thus

$$\frac{d}{d\tau}(m\dot{R}) = \frac{m_{,R}\dot{R}^2}{2} - \frac{1}{2} \left\{ m\Phi^- - m \left[ \frac{\Delta\Phi - m^2/R^2}{2m/R} \right]^2 \right\}_{,R},$$

where “ $,R$ ” means the derivative with respect to  $R$ .

Using time symmetry we can easily decrease an order of this differential equation and obtain Eq. (6) up to additive constant which can be calibrated to zero (note, it is zero only on trajectories thus it is a constraint). Therefore our Lagrangian indeed describes dynamics of thin shells. The moment conjugate to the variable  $R$  is  $\Pi = m\dot{R}$ , and the (super)Hamiltonian is

$$H = \frac{\Pi^2}{2m} + \frac{m}{2} \left\{ \Phi^- - \left[ \frac{\Delta\Phi - m^2/R^2}{2m/R} \right]^2 \right\}. \quad (8)$$

The prefix “super” means that, strictly speaking,  $H$  is the functional defined on the superspace which is the space of all worldsheet 3-metrics (4) and matter field configurations acting on a shell. Only due to spherical symmetry and presence of a single non-propagating degree of freedom we can obtain it as a standard Hamiltonian.

Recalling the above-mentioned zero constant we obtain that  $H = 0$  on the trajectories (6). Thus we mean this (super)Hamiltonian as a constraint, i.e.

$$H \approx 0, \quad (9)$$

or, in the quantum case ( $\Pi = -i\partial_R$ )

$$H\Psi \approx 0, \quad (10)$$

and can directly quantize Eq. (6) without any assumption about a time as was done in the special case  $m = \text{const}$  (dust shell,  $\sigma = m/4\pi R^2$ ) in Ref. [11]. Therefore, one always can quantize Eq. (6) directly without redundant motivations about a time gauge etc., just replace  $\dot{R}^2$  by  $\Pi^2/m^2$ .

Then in Planckian units we obtain the wave equation for the Wheeler-DeWitt wave function  $\Psi(R)$ :

$$\Psi'' + m^2 \left\{ \left[ \frac{\Delta\Phi - m^2/R^2}{2m/R} \right]^2 - \Phi^- \right\} \Psi = 0, \quad (11)$$

from which (for bound states if they exist) we can obtain spectra for necessary values, e.g., total mass-energy  $M_+$  [12] etc.

However, it should be noted that there exists another Lagrangian.

(b) *Approach with Planckian mass*

Let us consider the minisuperspace model described by the Lagrangian

$$L = \frac{m_{\text{pl}} \dot{R}^2}{2} - \frac{m_{\text{pl}}}{2} \left\{ \Phi^- - \left[ \frac{\Delta\Phi - m^2/R^2}{2m/R} \right]^2 \right\}. \quad (12)$$

In the same manner as was pointed out above one can see that the appropriate equation of motion yields Eq. (6). The moment conjugate to the variable  $R$  is  $\Pi = m_{\text{pl}} \dot{R}$  and the (super)Hamiltonian is

$$H = \frac{\Pi^2}{2m_{\text{pl}}} + \frac{m_{\text{pl}}}{2} \left\{ \Phi^- - \left[ \frac{\Delta\Phi - m^2/R^2}{2m/R} \right]^2 \right\}. \quad (13)$$

It is easy to see that Eqs. (9), (10) are valid. Therefore, one can quantize Eq. (6) directly by means of changing  $\dot{R}^2$  by  $\Pi^2/m_{\text{pl}}^2$ . In Planckian units we therefore obtain the wave equation for the Wheeler-DeWitt wave function  $\Psi(R)$ :

$$\Psi'' + \left\{ \left[ \frac{\Delta\Phi - m^2/R^2}{2m/R} \right]^2 - \Phi^- \right\} \Psi = 0. \quad (14)$$

Comparing Eqs. (11) and (14), we see that they are not the same and lead to different results in general case; however, we can not give absolute preference to any of them. Moreover, our Lagrangians are defined always up to some arbitrary multiplicative function of  $R$  which does not affect on the equation of motion (6) but necessarily appears in Hamiltonians. The arbitrariness of this function is nothing but the arbitrariness of the choice of an appropriate gauge.

Finally it should be pointed out how the wormhole/blackhole topology should be taken into account at quantization. Indeed from Eqs. (6), (8), (13) it is evident that by double squaring we annihilated the root signs  $\epsilon$  which determine topology. However at quantization we always can take into account topology because Eqs. (11), (14) should be supplemented by boundary conditions (e.g., at zero and spatial infinity) which are evidently determined by a specific (wormhole or black hole) topology.

Thus in present paper we worked out the minisuperspace approach and performed nonperturbative canonical quantization of spherically symmetric singular hypersurfaces in general relativity.

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